Gravitational 2-body problem without approximations: I

- Conserved energy
- two tricks:
  - → θ as independent variable
  - → $u = 1/r$ as new dependent var.
- orbit equation

Recall the Lagrangian for the two variables, r and θ, that describe the relative separation of the two masses:

$$L = \tfrac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{A}{r}$$

$$\mu = \frac{M_1 M_2}{M_1 + M_2}, \quad A = G M_1 M_2$$

Since t is absent from L, we know that H is conserved. Moreover, because T is quadratic in the velocities we know that H is just $T+V$:

$$H = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{A}{r}$$
$$= E \; (= \text{constant in time})$$

Because the momentum conjugate to θ,

$$L_z = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2 \dot{\theta},$$

is also conserved, we can write down an expression for E that involves only $\dot{r}$ and r:

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{L_z^2}{2\mu r^2} - \frac{A}{r}$$

We could solve this for $\dot{r}$ and then express t as an integral involving a function of r. This still leaves open the mystery we have seen for the perturbed circular orbit that the time-dependence of r stays "in synch" with the time-dependence of θ. So to address this mystery head-on, we will change the independent variable in the r-equation from t to θ:

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{dr}{d\theta}\cdot\frac{L_z}{\mu r^2}$$

The conserved E is now

$$E = \frac{1}{2}\frac{L_z^2}{\mu r^4}\left(\frac{dr}{d\theta}\right)^2 + \frac{L_z^2}{2\mu r^2} - \frac{A}{r}.$$

We can ~~further~~ simplify this by changing to a new dependent variable:

$$r(\theta) = 1/u(\theta)$$

$$\frac{dr}{d\theta} = -\frac{1}{u^2}\frac{du}{d\theta}$$

$$\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \left(\frac{du}{d\theta}\right)^2$$

$$E = \frac{L_z^2}{2\mu}\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] - Au$$

The final step in our solution method is to notice

$$0 = \frac{dE}{dt} = \frac{dE}{d\theta}\cdot\dot{\theta}$$

$$\Rightarrow 0 = \frac{dE}{d\theta} = \frac{L_z^2}{2\mu}\left[2\left(\frac{du}{d\theta}\right)\frac{d^2u}{d\theta^2} + 2u\frac{du}{d\theta}\right] - A\frac{du}{d\theta}$$

Dividing out the common factor of $\frac{du}{d\theta}$:

$$\frac{d^2u}{d\theta^2} + u = A\frac{\mu}{L_z^2} = \text{const.}$$

This <u>linear</u> differential equation is easily solved in complete generality:

$$u(\theta) = \left(\begin{array}{l}\text{arbitrary solution of} \\ \text{inhomogeneous equation}\end{array}\right) + \left(\begin{array}{l}\text{2-parameter general sol.} \\ \text{of homogeneous eqn.}\end{array}\right)$$

$$= \left(\frac{A\mu}{L_z^2}\right) + u_0 \cos(\theta - \theta_0)$$

We've chosen u_0 and θ_0 as our parameters for the general sol. because they have a simple interpretation.

Before we examine the sol. in detail, let's make note of the fact that $u(\theta)$ is a periodic function of θ with period 2π.

In other words, $r(\theta) = 1/u(\theta)$ repeats exactly after the angle of the orbit has changed by 2π: the orbit retraces itself! To see how this relates to the inverse-square nature of the gravitational force, suppose we had an "inverse-cube" law of gravity, ~~with~~ and the following change in our conserved energy:

$$E = T - \frac{A}{r^2}$$

$$= T - Au^2$$

Everything in our solution would go through as before, except that the differential equation for u will <u>not</u> have a constant term:

$$\frac{d^2u}{d\theta^2} + u = 2u\left(\frac{A\mu}{L_z^2}\right)$$

$$\frac{d^2u}{d\theta^2} + K^2u^2 = 0$$

$$K = \sqrt{1 - 2\frac{A\mu}{L_z^2}}$$

$$u = u_0 \cos K(\theta - \theta_0)$$

Now $r = 1/u$ has periodicity 2π only if K = integer, which need not be true since K varies continuously with L_z.

Let's write our solution (for the inverse-square law) in a more standard form with some definitions:

$$u(\theta) = \frac{1}{r(\theta)} = \frac{1}{r_0}\left(1 + \epsilon \cos(\theta - \theta_0)\right)$$

$$r_0 = \frac{L_z^2}{A\mu} \quad , \quad \epsilon = \frac{u_0}{A\mu / L_z^2}$$

While r_0 is defined by the physical parameters A, μ and the conserved angular momentum, we can let the "eccentricity" ϵ replace the arbitrary ~~positive~~ parameter u_0.

Thus:

$$r(\theta) = \frac{r_0}{1 + \epsilon \cos(\theta - \theta_0)}$$